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Certain Subclasses of Bi-Univalent Functions Defined by $(p, q)-$Differential Operator

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Abstract

The attention of the current paper, is to discover a new subclass $\mathcal{M}_{p,q}(\zeta, \Psi)$ and $\mathcal{F}_{p,q}(\delta, \Psi)$ of bi-univalent functions in the open disc $\mathfrak{U}$, using $(p, q)-$differential operator. Furthermore, we perceive estimates on coefficients $|x_2|$ and $|x_3|$ for functions in the new subclass.

Keywords: analytic function; univalent function; bi-univalent function; starlike function; convex function; $(p, q)-$derivative operator.

1 Introduction

Let $\mathscr{A}$ be a sign of the class of functions of the structure,

$$\mathscr{L}(\varsigma) = \varsigma + \sum_{d=2}^{\infty} a_d \varsigma^d, \tag{1}$$

which are analytic in the open unit disc $\mathfrak{U} = \{\varsigma : |\varsigma| < 1\}$. Also, $\mathscr{S}$ indicates the set of all subclasses of $\mathscr{A}$ that are schlit or univalent in $\mathfrak{U}$. In spite of the inverse functions of single-value functions are inverse functions, they don't need to be defined on the whole unit disc $\mathfrak{U}$. Obviously, according to the $K\ddot{o}ebe's$ one-quarter theorem [4], which states that the disc with radius $\frac{1}{4}$ is the co-domain of $\mathfrak{U}$. As a result, each and every univalent function $\mathscr{L}$ has an inverse $\mathscr{L}^{-1}$ satisfying,

$$\mathscr{L}^{-1}(\mathscr{L}(\varsigma)) = \varsigma, \big(\varsigma \in \mathfrak{U}\big), \quad \text{and} \quad \mathscr{H}(w) = \mathscr{L}^{-1}(\mathscr{L}(w)) = w, \quad \left(|w| < r_0(\mathscr{L}); r_0(\mathscr{L}) \geq \frac{1}{4}\right),$$

where

$$\mathscr{L}^{-1}(w) = w - x_2 w^2 + (2x_2^2 - x_3)w^3 - (5x_2^3 - 5x_2 x_3 + x_4)w^4 + \dots. \tag{2}$$

A function $\mathscr{L} \in \mathscr{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both $\mathscr{L}(\varsigma)$ and $\mathscr{L}^{-1}(\varsigma)$ are univalent in $\mathfrak{U}$. Let Σ denote the class of bi-univalent functions in $\mathfrak{U}$ given by (1). For a summary of the facts history and compulsive examples of functions in the class Σ, various writers have established and demonstrated subclasses of Σ.

The classes $\mathscr{H}_{\Sigma}^{\alpha}(0 < \alpha \leqq 1)$ and $\Sigma^*(\alpha)$ of bi-univalent functions were introduced by Srivastava et al. [13] and Brannan et al. [1] respectively and they showed for every function $f \in \Sigma$ of the form (1). Lewin [8] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Hadi et al. [6] have concerned in examining of bi-univalent functions and investigated the bounds of the coefficients estimate $|a_2|$ and $|a_3|$ by using q-convolution operator.

For two analytic functions in the unit disc $\mathfrak{U}$, then an analytic function $\mathscr{L}$ is subordinate to an analytic function $\mathscr{G}$, written $\mathscr{L}(\varsigma) \prec \mathscr{G}(\varsigma)$ if $\mathscr{L}$ can be indicate in form of composition of $\mathscr{G}$ and w as $\mathscr{L}(\varsigma) = \mathscr{G}(w(\varsigma))$ subject to the existance of analytic function w that satisfies the following condition $w(0) = 0$ and $|w(\varsigma)| < |(\varsigma)|$.

In their work, Ma and Minda [10] introduced a unification of different subclasses of starlike and convex functions. They considered an analytic function Ψ defined on the domain $\mathfrak{U}$, where $\mathfrak{U}$ is a unit disc. The function Ψ satisfies $\Psi(0) = 1$ and $\Psi^{'}(0) > 0$. In addition, it is assumed that Ψ maps the unit disk onto a region that is starlike with respect to a specific point and symmetric with respect to the real axis. In the follow-up, it is assumed that a function of this kind can be represented by a series expansion in the structure,

$$\Psi(\varsigma) = 1 + X_1\varsigma + X_2\varsigma^2 + X_3\varsigma^3 + \cdots (X_1 > 0). \tag{3}$$

The concept of strongly starlike functions is a topic in complex analysis and typically deals with starlike functions with respect to some fixed point. In this case, you have mentioned a function,

$$\Psi(\varsigma) = \left(\frac{1+\varsigma}{1-\varsigma}\right)^{\gamma} = 1 + 2\gamma\varsigma + 2\gamma^2\varsigma^2 + \cdots (0 < \gamma \leq 1), \tag{4}$$

and want to determine if it belongs to the class of strongly starlike functions of order $\gamma(0 < \gamma \leq 1)$, which provides $X_1 = 2\gamma$ and $X_2 = 2\gamma^2$.

The quantum calculus has numerous applications in the fields of special functions and many further spaces. We utilize essential notations and $(p, q)-$calculus is handed by Chakrabarti and Jagannathan [2],

$$[d]_{p,q} = \frac{p^d - q^d}{p - q}, \qquad p > 0, \qquad q > 0.$$

In particular, we attain,

$$\lim_{p \to 1} [d]_{p,q} = [d]_q,$$

which studied by Kac and Cheung [7].
Obviously, the notation is symmetric, i.e,

$$[d]_{p,q} = [d]_{q,p}.$$

Note that, Milovanović et al. [11] and Gupta[5] established some basic properties of $(p, q)-$operators.

Definition 1.1. *The $(p, q)-$derivative operator of a function $\mathscr{L}$ is defined by,*

$$D_{p,q}\mathscr{L}(\varsigma) = \frac{\mathscr{L}(p\varsigma) - \mathscr{L}(q\varsigma)}{\varsigma(p - q)}, \qquad (p \neq q, \quad \varsigma \in \mathfrak{U}), \tag{5}$$

$$D_{p,q}\mathscr{L}(\varsigma) = 1 + \sum_{d=2}^{\infty} [d]_{p,q} a_d \varsigma^{d-1}, \tag{6}$$

in addition we remark that $\lim\limits_{q \longrightarrow 1^-} \lim\limits_{p \longrightarrow 1^-} D_{p,q}\mathscr{L}(\varsigma) = \mathscr{L}^{'}(\varsigma),$

$$\big(D_{p,q}\mathscr{L}\big)(0) = \mathscr{L}^{'}(0),$$

provided that $\mathscr{L}$ is differentiable at 0.

2 Bi-Univalent Function Class $\mathcal{M}_{p,q}(\zeta, \Psi)$

We introduce a subclass $\mathcal{M}_{p,q}(\zeta, \Psi)$ of Σ and find approximate on the coefficient $|x_2|$ and $|x_3|$ for the functions in the recently developed subclass, by subordination.

Definition 2.1. *A function $\mathscr{L}$ in the form* (1) *belongs to the class $\mathcal{M}_{p,q}(\zeta, \Psi)$ if it satisfies the following subordination conditions,*

$$(1 - \zeta)\left(\frac{\varsigma D_{p,q}\mathscr{L}(\varsigma)}{\mathscr{L}(\varsigma)}\right) + \zeta\left(\frac{D_{p,q}\big(\varsigma D_{p,q}\mathscr{L}(\varsigma)\big)}{D_{p,q}\mathscr{L}(\varsigma)}\right) \prec \Psi(\varsigma), \qquad (0 \leq \zeta \leq 1, 0 < p < 1, 0 < q < 1), \tag{7}$$

and

$$(1 - \zeta)\left(\frac{w D_{p,q}\mathscr{H}(w)}{\mathscr{H}(w)}\right) + \zeta\left(\frac{D_{p,q}\big(w D_{p,q}\mathscr{H}(w)\big)}{D_{p,q}\mathscr{H}(w)}\right) \prec \Psi(w), \qquad (0 \leq \zeta \leq 1, 0 < p < 1, 0 < q < 1), \tag{8}$$

where ς, w are within $\mathfrak{U}$, and $\mathscr{H}$ is defined as in (2).

Remark 2.1.

1. *For $p \to 1^-$, $q \to 1^-$, the class $\mathcal{M}_{p,q}(\zeta, \Psi)$ reduces the class $\mathcal{M}_(\gamma, \zeta)$ studied by Li and Wang [9].*
2. *For $p \to 1^-$, $q \to 1^-$ and $\zeta = 0$, the class $\mathcal{M}_{p,q}(\zeta, \Psi)$ reduces to the class $\mathscr{S}_\Sigma(\lambda, \gamma, \varphi)$ which introduced by Deniz [3].*
3. *For $p \to 1^-$, $\gamma = 0$, $\lambda = 1$ and different parameters $r, \mathfrak{s}, \mathfrak{t}$, the class investigated by Srivastava et al. [14] reduces to the class $\mathcal{M}_q(\zeta, \Psi)$, for $p \to 1^-, \sigma = 1, \delta = 0$ and some parameters of $\mathfrak{t}, n$ we have the class studied by Wanas and Mahdi [15].*

We require this lemma to establish the main result.

Lemma 2.1. [12] *Let $\mathfrak{T}$ be the family of all functions $\mathfrak{S}$ that are analytic in $\mathfrak{U}$ with $\mathfrak{S}(0) = 1$ and $\Re(\mathfrak{S}(\varsigma)) > 0 (\forall \varsigma \in \mathfrak{U})$. If a function $\mathfrak{S} \in \mathfrak{T}$ is given by $\mathfrak{S}(\varsigma) = 1 + r_1\varsigma + r_2\varsigma^2 + \ldots$ for ς in the unite disk then $|r_k| \leq 2 \quad (\forall\, k \in \mathbb{N})$.*

3 Main Result

Theorem 3.1. *Let $\mathscr{L}(\varsigma)$ given by (1) be in the class $\mathcal{M}_{p,q}(\zeta, \Psi)$. Then,*

$$|x_2| \leq \frac{X_1\sqrt{X_1}}{\sqrt{\left|\Big((Z-1)(1-\zeta+Z\zeta) - (Y-1)(1-\zeta+Y^2\zeta)\Big)X_1^2 + (Y-1)^2(1-\zeta+Y\zeta)^2(X_1 - X_2)\right|}}, \tag{9}$$

and

$$|x_3| \leq \left(\frac{X_1}{(Y-1)(1-\zeta+Y\zeta)}\right)^2 + \left(\frac{X_1}{(Z-1)(1-\zeta+Z\zeta)}\right), \tag{10}$$

where $0 \leq \zeta \leq 1$, $Y = [2]_{p,q}$ and $Z = [3]_{p,q}$.

Proof. Consider $\mathscr{L} \in \mathcal{M}_{p,q}(\zeta, \Psi)$ and $\mathscr{L}^{-1} = \mathscr{H}$. In this scenario, an analytic function exists $\mathcal{F}, \mathcal{G} : \mathfrak{U} \to \mathfrak{U}$ along $\mathcal{F}(0) = 0 = \mathcal{G}(0)$, satisfying,

$$(1-\zeta)\left(\frac{\varsigma D_{p,q}\mathscr{L}(\varsigma)}{\mathscr{L}(\varsigma)}\right) + \zeta\left(\frac{D_{p,q}\big(\varsigma D_{p,q}\mathscr{L}(\varsigma)\big)}{D_{p,q}\mathscr{L}(\varsigma)}\right) = \Psi\big(\mathcal{F}(\varsigma)\big), \tag{11}$$

and

$$(1-\zeta)\left(\frac{wD_{p,q}\mathscr{H}(w)}{\mathscr{H}(w)}\right) + \zeta\left(\frac{D_{p,q}\big(wD_{p,q}\mathscr{H}(w)\big)}{D_{p,q}\mathscr{H}(w)}\right) = \Psi\big(\mathcal{G}(w)\big). \tag{12}$$

Define the functions $\eta(\varsigma)$ and $\mathcal{J}(\varsigma)$ as follows;

$$\eta(\varsigma) := \frac{1+\mathcal{F}(\varsigma)}{1-\mathcal{F}(\varsigma)} = 1 + \eta_1\varsigma + \eta_2\varsigma^2 + \ldots,$$

and

$$\mathcal{J}(\varsigma) := \frac{1+\mathcal{G}(\varsigma)}{1-\mathcal{G}(\varsigma)} = 1 + \mathcal{J}_1\varsigma + \mathcal{J}_2\varsigma^2 \dots .$$

Alternatively,

$$\mathcal{F}(\varsigma) := \frac{\eta(\varsigma)-1}{\eta(\varsigma)+1} = \frac{1}{2}\Big[\eta_1\varsigma + \Big(\eta_2 - \frac{\eta_1^2}{2}\Big)\varsigma^2 + \dots\Big], \tag{13}$$

$$\mathcal{G}(\varsigma) := \frac{\mathcal{J}(\varsigma)-1}{\mathcal{J}(\varsigma)+1} = \frac{1}{2}\Big[\mathcal{J}_1\varsigma + \Big(\mathcal{J}_2 - \frac{\mathcal{J}_1^2}{2}\Big)\varsigma^2 + \dots\Big]. \tag{14}$$

Both $\eta(\varsigma)$ and $\mathcal{J}(\varsigma)$ are analytic within $\mathfrak{U}$ and share the initial values $\eta(0) = 1$ and $\mathcal{J}(0) = 1$. Since $\mathcal{G}$ and $\mathcal{F}$ are functions that map from $\mathfrak{U}$ to $\mathfrak{U}$, $\eta(\varsigma)$ and $\mathcal{J}(\varsigma)$ exhibit a real part that is greater than zero when evaluated in $\mathfrak{U}$, in which$|\mathcal{J}_k| \le 2$, $|\eta_k| \le 2$.

By substituting (13) into (11) and (14) into (12), we obtain

$$(1-\zeta)\left(\frac{\varsigma D_{p,q}\mathscr{L}(\varsigma)}{\mathscr{L}(\varsigma)}\right) + \zeta\left(\frac{D_{p,q}\big(\varsigma D_{p,q}\mathscr{L}(\varsigma)\big)}{D_{p,q}\mathscr{L}(\varsigma)}\right) = \Psi\left(\frac{1}{2}\Big[\eta_1\varsigma + \Big(\eta_2 - \frac{\eta_1^2}{2}\Big)\varsigma^2 + \dots\Big]\right), \tag{15}$$

and

$$(1-\zeta)\left(\frac{wD_{p,q}\mathscr{H}(w)}{\mathscr{H}(w)}\right) + \zeta\left(\frac{D_{p,q}\big(wD_{p,q}\mathscr{H}(w)\big)}{D_{p,q}\mathscr{H}(w)}\right) = \Psi\left(\frac{1}{2}\Big[\mathcal{J}_1\varsigma + \Big(\mathcal{J}_2 - \frac{\mathcal{J}_1^2}{2}\Big)\varsigma^2 + \dots\Big]\right). \tag{16}$$

In view of (1), (2) and from (15) and (16), we obtain

$$\begin{aligned}1 &+ \Big[\big(1-\zeta+Y\zeta\big)\big(Y-1\big)\Big]x_2\varsigma - \Big[\Big(\big(Y-1\big)\big(\big(1-\zeta+Y^2\zeta\big)\Big)x_2^2 + \Big(\big(1-\zeta+Z\zeta\big)\big(Z-1\big)\Big)x_3\Big]\varsigma^2 \\ &+ \dots = 1 + \frac{1}{2}X_1\eta_1\varsigma + \Big[\frac{1}{2}X_1\big(\eta_2 - \frac{\eta_1^2}{2}\big) + \frac{1}{4}X_2\eta_1^2\Big]\varsigma^2 + \cdots ,\end{aligned}$$

and

$$\begin{aligned}1 &- \Big[\big(1-\zeta+Y\zeta\big)\big(Y-1\big)\Big]x_2w \\ &+ \left(\Big[2\big(Z-1\big)\big(1-\zeta+Z\zeta\big) - \big(1-\zeta+Y^2\zeta\big)\big(Y-1\big)\Big]x_2^2 - \Big[\big(Z-1\big)\big(1-\zeta+Z\zeta\Big]x_3\right)w^2 \\ &= 1 + \frac{1}{2}X_1\mathcal{J}_1w + \Big[\frac{1}{2}X_1\big(\mathcal{J}_2 - \frac{\mathcal{J}_1^2}{2}\big) + \frac{1}{4}X_2\mathcal{J}_1^2\Big]w^2 + \cdots ,\end{aligned}$$

these relationships are as follows:

$$\big(Y-1\big)\big(1-\zeta+Y\zeta\big)x_2 = \frac{1}{2}X_1\eta_1, \tag{17}$$

$$-\Big[\big(Y-1\big)\big(1-\zeta+Y^2\zeta\big)\Big]x_2^2 + \Big[\big(Z-1\big)\big(1-\zeta+Z\zeta\big)\Big]x_3 = \frac{1}{2}X_1\big(\eta_2 - \frac{\eta_1^2}{2}\big) + \frac{1}{4}X_2\eta_1^2, \tag{18}$$

$$-\Big[\big(Y-1\big)\big(1-\zeta+Y\zeta\big)\Big]x_2 = \frac{1}{2}X_1\mathcal{J}_1, \tag{19}$$

$$\begin{aligned}&\Big[2\big(Z-1\big)\big(1-\zeta+Z\zeta\big) - \big(Y-1\big)\big(1-\zeta+Y^2\zeta\big)\Big]x_2^2 \\ &\qquad - \Big[\big(Z-1\big)\big(1-\zeta+Z\zeta\big)\Big]x_3 = \frac{1}{2}X_1\big(\mathcal{J}_2 - \frac{\mathcal{J}_1^2}{2}\big) + \frac{1}{4}X_2\mathcal{J}_1^2.\end{aligned} \tag{20}$$

From (17) and (19), we have

$$\eta_1 = -\mathcal{J}_1, \tag{21}$$

and

$$8\big(Y-1\big)^2\big(1-\zeta+Y\zeta\big)^2 x_2^2 = X_1^2\big(\eta_1^2+\mathcal{J}_1^2\big). \tag{22}$$

From (18) and (20) and (22),

$$x_2^2 = \frac{X_1^3(\eta_2+\mathcal{J}_2)}{4\Bigg(\Big[(Z-1)(1-\zeta+Z\zeta)-(Y-1)(1-\zeta+Y^2\zeta)\Big]X_1^2+(Y-1)^2(1-\zeta+Y\zeta)^2(X_1-X_2)\Bigg)}.$$

By utilizing Lemma 2.1 for the coefficients η_2 and $\mathcal{J}_2$, we obtain

$$|x_2| \leq \frac{X_1\sqrt{X_1}}{\sqrt{\Big|((Z-1)(1-\zeta+Z\zeta)-(Y-1)(1-\zeta+Y^2\zeta))X_1^2+(Y-1)^2(1-\zeta+Y\zeta)^2(X_1-X_2)\Big|}}.$$

By subtracting (20) from (18) and using (21) and (22), we obtain

$$x_3 = \frac{X_1^2\big(\eta_1^2+\mathcal{J}_1^2\big)}{8\big(Y-1\big)^2\big(1-\zeta+Y\zeta\big)^2} + \frac{X_1\big(\eta_2-\mathcal{J}_2\big)}{4\big(Z-1\big)\big(1-\zeta+Z\zeta\big)}.$$

Using Lemma 2.1 once more for the coefficients of $\eta_1, \eta_2, \mathcal{J}_1, \mathcal{J}_2$, we have

$$|x_3| \leq \left(\frac{X_1}{\big(Y-1\big)\big(1-\zeta+Y\zeta\big)}\right)^2 + \frac{X_1}{\big(Z-1\big)\big(1-\zeta+Z\zeta\big)}.$$

□

For $p \to 1^-$, $q \to 1^-$ and $X_1 = 2\gamma$, $X_2 = 2\gamma^2$, Theorem 3.1 proceeds the following corollary.

Corollary 3.1. [9] *When function $\mathscr{L}$, as expressed in equation (1), is part of the $\mathcal{M}_(\gamma,\zeta)$ class, then,*

$$|x_2| \leq \frac{2\gamma}{\sqrt{\big|(1+\zeta)(\gamma+1+\zeta-\zeta\gamma)\big|}}, \tag{23}$$

and

$$|x_3| \leq \frac{4\gamma^2}{(1+\zeta)^2} + \frac{\gamma}{1+2\zeta}. \tag{24}$$

4 Bi-Univalent Function Class $\mathcal{F}_{p,q}(\delta,\Psi)$

Definition 4.1. *A function $\mathscr{L} \in \Sigma$ of the form (1) is said to be in the class $\mathcal{F}_{p,q}(\zeta,\Psi)$ if the following subordination holds;*

$$(1-\delta)\left(\frac{\mathscr{L}(\varsigma)}{\varsigma}\right) + \delta\Big(D_{p,q}\mathscr{L}(\varsigma)\Big) \prec \Psi(\varsigma), \qquad (0 \leq \delta \leq 1, \quad 0 < p < 1, \quad 0 < q < 1), \tag{25}$$

and

$$(1-\delta)\left(\frac{\mathscr{H}(w)}{w}\right)+\delta\left(D_{p,q}\mathscr{H}(w)\right)\prec\Psi(w),\qquad(0\leq\delta\leq1,\quad 0<p<1,\quad 0<q<1),\tag{26}$$

where $\varsigma, w\in\mathfrak{U}$, *and* $\mathscr{H}$ *is given by* (2).

Theorem 4.1. *Let* $\mathscr{L}$ *given by* (1) *be in the class* $\mathcal{F}_{p,q}(\delta,\Psi)$. *Then,*

$$|x_2|\leq\frac{X_1\sqrt{X_1}}{\sqrt{\left|(1-\delta+Z\delta)X_1^2+(1-\delta+Y\delta)^2(X_1-X_2)\right|}},\tag{27}$$

and

$$|x_3|\leq X_1\left(\frac{1}{(1-\delta+Z\delta)}\right)+\left(\frac{X_1}{(1-\delta+Y\delta)}\right)^2,\qquad\text{where } Z=[3]_{p,q}\text{ and } Y=[2]_{p,q}.\tag{28}$$

Proof. By following the same steps as in the proof of Theorem 3.1, we obtain the following relationship;

$$(1-\delta+Y\delta)x_2=\frac{1}{2}X_1\eta_1,\tag{29}$$

$$(1-\delta+Z\delta)x_3=\frac{1}{2}X_1(\eta_2-\frac{\eta_1^2}{2})+\frac{1}{4}X_2\eta_1^2,\tag{30}$$

$$-(1-\delta+Y\delta)x_2=\frac{1}{2}X_1\mathcal{J}_1,\tag{31}$$

$$2(1-\delta+Z\delta)x_2^2-(1-\delta+Z\delta)x_3=\frac{1}{2}X_1(\mathcal{J}_2-\frac{\mathcal{J}_1^2}{2})+\frac{1}{4}X_2\mathcal{J}_1^2.\tag{32}$$

From (29) and (31), we get

$$\eta_1=-\mathcal{J}_1,\tag{33}$$

$$8(1-\delta+Y\delta)^2x_2^2=X_1^2(\eta_1^2+\mathcal{J}_1^2).\tag{34}$$

Additionally, by examining (30), (32) and (34), we discover that,

$$x_2^2=\frac{X_1^3(\eta_2+\mathcal{J}_2)}{4\left[(1-\delta+Z\delta)X_1^2+(1-\delta+Y\delta)^2(X_1-X_2)\right]}.$$

By employing Lemma 2.1 for the coefficients η_2 and $\mathcal{J}_2$, we promptly obtain,

$$|x_2|\leq\frac{X_1\sqrt{X_1}}{\sqrt{\left|(1-\delta+Z\delta)X_1^2+(1-\delta+Y\delta)^2(X_1-X_2)\right|}}.$$

This establish the bound on $|x_2|$ as presented in (27). To express bound on $|x_3|$, by subtracting (32) from (30) and using (34) we get,

$$x_3=\frac{X_1^2(\eta_1^2+\mathcal{J}_1^2)}{8(1-\delta+Y\delta)^2}+\frac{X_1(\eta_2-\mathcal{J}_2)}{4(1-\delta+Z\delta)}.$$

After applying Lemma 2.1 for the coefficients η_1, η_2, $\mathcal{J}_1$ and $\mathcal{J}_2$, we get,

$$|x_3| \leq X_1\left(\frac{1}{(1-\delta+Z\delta)}\right)+\left(\frac{X_1}{(1-\delta+Y\delta)}\right)^2. \tag{35}$$

This completes the proof of Theorem 4.1. □

5 Conclusions

In this article, we have presented a novel, subclasses of Σ defined $(p, q)-$differential operator. We even found the upper bounds for the coefficients $|x_2|$ and $|x_3|$ for the functions that belong to this original subclass and its subclasses.

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References

[1] D. A. Brannan, J. Clunie & W. E. Kirwan (1970). Coefficient estimates for a class of starlike functions. *Canadian Journal of Mathematics*, 22(3), 476–485. https://doi.org/10.4153/CJM-1970-055-8.

[2] R. Chakrabarti & R. Jagannathan (1991). A (p, q)-oscillator realization of two-parameter quantum algebras. *Journal of Physics A: Mathematical and General*, *24*(13), Article ID: L711. https://dx.doi.org/10.1088/0305-4470/24/13/002.

[3] E. Deniz (2013). Certain subclasses of bi-univalent functions satisfying subordinate conditions. *Journal of Classical Analysis*, 2(1), 49–60. http://dx.doi.org/10.7153/jca-02-05.

[4] P. Duren (1994). *Linear and Complex Analysis Problem Book 3: Part II*, volume 1574, chapter Geometric Function Theory, pp. 383–422. Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0101068.

[5] V. Gupta (2016). (p, q)-Baskakov-Kantorovich operators. *Applied Mathematics & Information Sciences*, *10*(4), 1551–1556. http://dx.doi.org/10.18576/amis/100433.

[6] S. H. Hadi, M. Darus & T. Bulboaca (2023). Bi-univalent functions of order ζ connected with (m, n)-Lucas polynomial. *Journal of Mathematics and Computer Science*, *31*(4), 433–447. http://dx.doi.org/10.22436/jmcs.031.04.06.

[7] V. Kac & P. Cheung (2002). *Quantum Calculus* volume 113. Springer, New York. https://doi.org/10.1007/978-1-4613-0071-7.

[8] M. Lewin (1967). On a coefficient problem for bi-univalent functions. *Proceedings of the American Mathematical Society*, *18*(1), 63–68. https://doi.org/10.2307/2035225.

[9] X. F. Li & A. P. Wang (2012). Two new subclasses of bi-univalent functions. *International Mathematical Forum*, *7*(30), 1495–1504.

[10] W. Ma & D. Minda (1992). A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis, 1992*, pp. 157–169. Nankai Institute of Mathematics. International Press. https://www.researchgate.net/publication/245129813_A_unified_treatment_of_some_special_classes_of_functions.

[11] G. V. Milovanović, V. Gupta & N. Malik (2018). (p, q)-Beta functions and applications in approximation. *Boletín de la Sociedad Matemática Mexicana*, *24*(1), 219–237. https://doi.org/10.1007/s40590-016-0139-1.

[12] C. Pommerenke (1975). *Univalent Functions*. Studia Mathematica: Mathematische Lehrbücher. Vandenhoeck and Ruprecht, Paderborn.

[13] H. M. Srivastava, A. K. Mishra & P. Gochhayat (2010). Certain subclasses of analytic and bi-univalent functions. *Applied Mathematics Letters*, *23*(10), 1188–1192. https://doi.org/10.1016/j.aml.2010.05.009.

[14] H. M. Srivastava, A. K. Wanas & R. Srivastava (2021). Applications of the q-Srivastava-Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials. *Symmetry*, *13*(7), Article ID: 1230. https://doi.org/10.3390/sym13071230.

[15] A. K. Wanas & A. M. Mahdi (2023). Applications of the q-Wanas operator for a certain family of bi-univalent functions defined by subordination. *Asian-European Journal of Mathematics*, *16*(6), Article ID: 2350095. http://dx.doi.org/10.1142/S179355712350095X.